

**Math 243  
Summer 2018  
Practice Exam 2  
Doomsday  
Time Limit: Probably Not Enough**

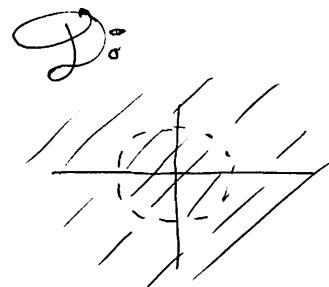
Name (Print): KEY

Problem	Points	Score
1	15	
2	20	
3	20	
4	15	
5	15	
6	20	
7	20	
8	20	
Total:	145	

1. (15 points) Let  $f(x) = \frac{1}{16-x^2-y^2}$

a) Find the domain and range of  $f(x)$ .

Domain:  $16 - x^2 - y^2 = 0$  when  $16 = x^2 + y^2$ .



Range: Kinda tricky, but you can show that the range is  $(-\infty, 0) \cup [\frac{1}{16}, \infty)$ .

b) Is the domain open/closed or neither? What is the boundary of the domain? Is the domain bounded or unbounded?

The domain is open, and unbounded, and the boundary of the domain is the circle  $x^2 + y^2 = 16$

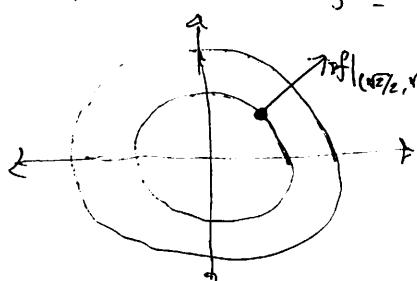
c) Graph the level curves  $f(x) = \frac{1}{\sqrt{15}}$  and  $f(x) = 5$ . Include the vector  $\nabla f|_{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$  on the appropriate level curve.

So.... there was a bit of a typo here...

The function was supposed to be  $f(x,y) = \frac{1}{\sqrt{16-x^2-y^2}}$  (so let's assume I wrote it this way)

In this case,  $\frac{1}{\sqrt{15}} = \frac{1}{\sqrt{16-x^2-y^2}} \Rightarrow x^2 + y^2 = 1$  (The point  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$  is on this curve)

and  $5 = \frac{1}{\sqrt{16-x^2-y^2}} \Rightarrow x^2 + y^2 = 16 - \frac{1}{25}$



$$\begin{aligned} \nabla f &= \frac{1}{2} (16 - x^2 - y^2)^{-3/2} (-2x) \mathbf{i} \\ &\quad + \frac{1}{2} (16 - x^2 - y^2)^{-3/2} (-2y) \mathbf{j} \end{aligned}$$

2. (a) (10 points) Find  $\lim_{(x,y) \rightarrow (2,2)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$  if it exists.

$$\begin{aligned} & \lim_{(x,y) \rightarrow (2,2)} \frac{x-y+2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} \cdot \frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (2,2)} \frac{(x-y)(\sqrt{x}+\sqrt{y}) + 2(x-y)}{x-y} \\ &= \lim_{(x,y) \rightarrow (2,2)} \frac{(x-y)[\sqrt{x}+\sqrt{y} + 2]}{x-y} \\ &= \sqrt{2} + \sqrt{2} + 2 \end{aligned}$$

(b) (10 points) Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4-y^2}{x^4+y^2}$  if it exists.

Along  $y = mx^2$ ,  $\frac{x^4-y^2}{x^4+y^2} = \frac{1-m^2}{1+m^2}$ ,  
and this depends on the value of  $m$ .  
Whence, the  $\lim$  DNE.

$$\begin{aligned}
 3. \text{ (a) (10 points) Find } & \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \frac{y^2 + \sin(y)}{y^4} + x \right) \right) \\
 &= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( \frac{y^2 + \sin(y)}{y^4} + x \right) \right) \\
 &= \frac{\partial}{\partial y} (1) \\
 &= 0
 \end{aligned}$$

(b) (10 points) Let  $f(x, y, z) = \frac{ye^{xyz}}{x}$ . Find  $f_x, f_y$  and  $f_z$ .

$$f_x = \frac{ye^{xyz}(yz) \cdot x - ye^{xyz}}{x^2}$$

$$f_y = \frac{1}{x} (e^{xyz} + ye^{xyz}(xz))$$

$$f_z = \frac{y}{x} e^{xyz} (xy)$$

$$= y^2 e^{xyz}$$

4. (15 points) a) Suppose that  $r(t) = g(t)i + h(t)j$  is a vector valued function such that  $f(g(t), h(t)) = c$  for some constant  $c$ . Show that  $\nabla f$  and  $\frac{dr}{dt}$  are orthogonal along this level curve.

$$\frac{d}{dt} (f(g(t), h(t))) = \frac{df}{dt}(c) \quad \text{and so}$$

$$\frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} = 0 \quad , \text{ but this is exactly}$$

$$\nabla f \cdot r'(t) = 0 \quad , \text{ and therefore}$$

$\nabla f$  and  $\frac{dr}{dt}$  are orthogonal!

- b) Find the derivative of  $f(x, y) = \ln(x^2 + y^2)$  in the direction of  $v = i + j$  at the point  $(1, 1)$ .

$$u = \frac{v}{\|v\|} = \frac{1}{\sqrt{2}}(i + j)$$

$$\nabla f = \frac{1}{x^2+y^2} (2x) i + \frac{1}{x^2+y^2} (2y) j$$

$$\begin{aligned} \nabla f|_{(1,1)} &= i + j \quad , \text{ so} \quad \nabla f \cdot u|_{(1,1)} = \frac{1}{\sqrt{2}}(1+1) \\ &= \frac{2}{\sqrt{2}} \end{aligned}$$

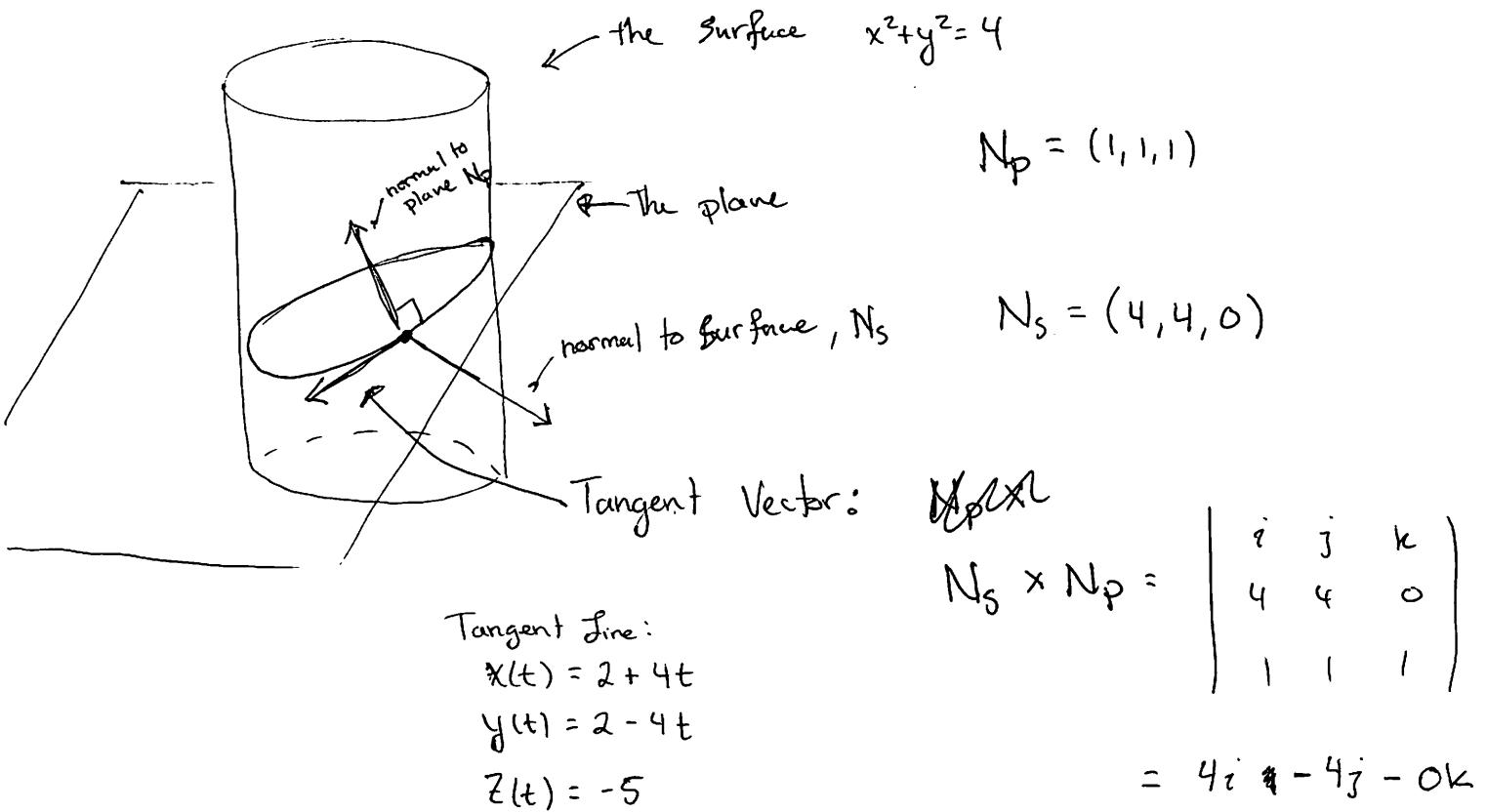
5. (15 points) a) Let  $z = x^2 - y^2 + 3$ . Find the equation of the tangent plane at the point  $(1, 1, 3)$ .

If  $f(x, y, z) = x^2 - y^2 - z + 3$  then the level surface  $f(x, y, z) = 0$   
is exactly this surface.

$$f_x = 2x, f_y = -2y, f_z = -1$$

$$2(x-1) - 2(y-1) - 1(z-3) = 0 \text{ is the tangent plane!}$$

- b) The surface  $x^2 + y^2 = 4$  is "sliced" by the plane  $x + y + z + 1 = 0$  and forms an ellipse. Find the parametric equations for the tangent line to this ellipse at the point  $(2, 2, -5)$ .



6. (a) (20 points) Let  $f(x, y) = 9x^3 + y^3/3 - 4xy$ . Use the second derivative test to find any local min/max or saddle points.

$$f_x = 27x^2 - 4y$$

$$f_y = y^2 - 4x$$

If  $y=0$  then  $x=0$

$$\boxed{(0, 0)}$$

If  $y = \frac{4}{3}$  then by (\*)

$$\frac{\left(\frac{4}{3}\right)^2}{4} = x$$

$$\text{so, } \frac{4}{9} = x$$

$$\boxed{\left(\frac{4}{9}, \frac{4}{3}\right)}$$

$$f_y = 0 \Rightarrow y^2 = 4x$$

$$\frac{y^2}{4} = x \quad (*)$$

Now, using  $f_x = 0$

$$\text{we get } \frac{27}{16}y^4 - 4y = 0$$

$$\text{or } \frac{3^3}{4^3}y^4 - y = 0$$

$$\text{so, } y\left(\frac{3^3}{4^3}y^3 - 1\right) = 0$$

$$\text{so } y=0 \text{ or } y = \frac{4}{3}$$

$$f_{xx} = 54x, \quad f_{xy} = -4$$

$$f_{yy} = 2y, \quad$$

$$f_{xx} f_{yy} - f_{xy}^2 = 108xy - 16 \quad \textcircled{1} \quad (0,0) \text{ this is } < 0$$

$$\textcircled{2} \quad \left(\frac{4}{9}, \frac{4}{3}\right) \text{ this is } > 0$$

+ + local min  $\textcircled{1} \quad \left(\frac{4}{9}, \frac{4}{3}\right)$

Saddle  $\textcircled{2} \quad (0, 0)$

7. (20 points) Find the point on the graph of  $z = xy + 1$  that is closest to the origin. (extra credit?)

$$f = x^2 + y^2 + z^2 - a^2 \quad \text{Constraint}$$

$$g(x, y, z) = xy - z + 1$$

$$\text{and } g = 0.$$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow 2x = \lambda y$$

$$2y = \lambda x$$

$$2z = \lambda(-1) \Rightarrow z = \frac{-\lambda}{2}$$

using  ~~$f = x^2 + y^2 + z^2 - a^2$~~   $y = \frac{\lambda x}{2}$

$\lambda = -2$  case:

$$x = -y \quad \text{and} \quad z = +1$$

$$\text{so, } 0 = x(-x) + 1 + 1$$

$$\cancel{x=0} \cancel{y=0} \cancel{z=0} \Rightarrow y=0.$$

$\lambda = 2$  case  $x = y \quad \text{and} \quad z = -1$

$0 = x^2 + 1 + 1$ , and this can't happen,

so,  $\lambda = -2$  and our point is

$$(0, 0, 1)$$

$$\begin{cases} 2x = \lambda \left(\frac{\lambda x}{2}\right) \\ x = \frac{\lambda^2 x}{4} \Rightarrow x \left(1 - \frac{\lambda^2}{4}\right) = 0 \end{cases}$$

$$\Rightarrow \frac{\lambda^2}{4} = 1 \quad \text{so}$$

$$\lambda = \pm 2$$

8. (20 points) Find the cubic approximation for the function  $f(x, y) = e^x \ln(1 + y)$  centered at the origin.

$$f(x, y) = \sum_n \sum_k a_{n,k} x^n y^k$$

where  $a_{n,k} = \frac{f_{x^n y^k}(0, 0)}{n! k!}$

$$f_x = e^x \ln(1+y)$$

$$f_{xy} = e^x \frac{1}{1+y}$$

$$f_y = e^x \frac{1}{1+y}$$

$$f_{xx} = e^x \ln(1+y)$$

$$f_{x^2y} = e^x \frac{1}{(1+y)^2}$$

$$f_{yy} = e^x \frac{-1}{(1+y)^2}$$

$$f_{xxx} = e^x \ln(1+y)$$

$$f_{xy^2} = e^x \frac{-1}{(1+y)^3}$$

$$f_{yyy} = e^x \frac{2}{(1+y)^3}$$

Cubic Approx.

$$f(x, y) \approx 0 + 0x + y + \frac{1}{2}xy + 0x^2 + \frac{1}{2}x^2y + \frac{-1}{2}y^2$$

$$+ 0x^3 + \frac{-1}{2}xy^2 + \frac{2}{6}y^3$$

$$= y + xy + \frac{1}{2}x^2y - \frac{1}{2}y^2 - \frac{1}{2}xy^2 + \frac{1}{3}y^3$$